

Coloring problem of signed interval graphs

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Abstract. The chromatic number of signed graphs is defined in [3] recently. The coloring and clique problem of interval graphs is considered in [4] and polynomial time algorithms are established. Here we consider these problems for signed interval graphs and prove that the coloring problem of signed interval graphs is NP-complete whereas their ordinary clique problem is in P. We also study the complexity of further related problems.

1 Introduction

In this paper we consider simple graphs $G = (V, E)$, i.e graphs with out loops and multiple edges. The complement of the graph G is denoted by G^c . For a vertex v of G , by $N_G(v)$ (or simply $N(v)$) we denote the set of neighbors of v in G and by $N_G[v]$ (or simply $N[v]$) we denote the closed neighborhood of v in G , that is, $N(v) \cup \{v\}$. For two graphs G and G' by $G \vee G'$ we mean the disjoint union of G and G' . An interval graph, which will be denoted by I is the intersection graph of a set of real intervals. Interval graphs have some characterization which make them easy to study. For instance in the following lemma from [1] a characterization of these graphs is established.

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Lemma 1 The graph I is an interval graph if and only if the maximal cliques of it can be ordered M_1, M_2, \dots, M_k such that for any i, j, k , where $i \leq j \leq k$, it has the property $M_i \cap M_k \subseteq M_j$.

The graph $G = (V, E)$ together with a function $s : E \longrightarrow \{1, -1\}$ on edges of G is a *signified graph*. We denote the corresponding signified graph with $\Sigma = (G, \sigma)$, where $\sigma = s^{-1}(-1)$. The graph G is called the ground graph of $\Sigma = (G, \sigma)$ and the set σ is called the signature of Σ . For any edge e of Σ , we call it a positive or negative edge if $s(e)$ has positive or negative sign respectively. The sets V, E are called the vertex set and edge set of Σ respectively. For a signified graph $\Sigma = (G, \sigma)$ by the *positive sub-graph* (*negative sub-graph*) we mean the graph on the vertex set $V(G)$ whose edges are the positive (negative) edges of Σ and is denoted by Σ^+ (Σ^-). Suppose that C is a cycle in G , the signature of C in (G, σ) is the product of signs of it's edges. A cycle is called *balanced* if it has positive signature, otherwise we call it *unbalanced*. A signified graph all of whose cycles are balanced is called balanced otherwise we call it unbalanced.

By *resigning* at a vertex v of Σ we mean to change sign of all the edges incident with v . Two signified graphs (G, σ) and (G, σ') are called *switching equivalent* if one is obtained from the other by a sequence of resignings. In [8], it has been proved that two signified graphs (G, σ) and (G, σ') are switching equivalent if and only if they have the same set of unbalanced cycles. Being switching equivalent is an equivalence relation on the set of all signified graphs over G . Any equivalence class of this relation will be called the *signed graph*. The equivalence class of signified graphs which are equivalent to (G, σ) will be denoted by $[G, \sigma]$. A signed graph $[H, \sigma']$ is called a signed sub graph of $[G, \sigma]$ if there are representations (H, σ'_1) and (G, σ_1) of them such that H is a sub graph of G and $\sigma'_1 \subseteq \sigma_1$. For a subset S of V , by $\langle S \rangle_\Sigma$ we denote the signed sub graph of (G, σ) which it's ground is the induced sub-graph of G on S . If there is no doubt about Σ then we simply write $\langle S \rangle$. For two subsets X, Y of V , by $\langle X, Y \rangle_\Sigma$ we denote the signed sub-graph of (G, σ) which the underlying graph is the bipartite sub-graph of G on the parties X, Y with all possible edges from X to Y . A proper coloring of a signified graph (G, σ) is a function $f : V \rightarrow \mathbb{N}$ such that if u is adjacent to v in G then $f(u) \neq f(v)$ also if there are two edges uv and $u'v'$ having the same pair of colors on their nodes then these two edges should have same sign. The smallest number k where there is a proper coloring of a signified graph (G, σ_1) in the class $[G, \sigma]$ with k colors is called the signed chromatic number of $[G, \sigma]$

and is denoted by $\chi_s[G, \sigma]$. A signed graph $[H, \sigma_0]$ on n vertices is called an S-clique if its signed chromatic number equals n . The S-clique number of $[G, \sigma]$ which is denoted by $w_s[G, \sigma]$ is the maximum size of a signed sub graph of $[G, \sigma]$ which is an S-clique. In this paper we consider the signed interval graphs, i.e the signed graphs which their ground is an interval graph, and prove that the problem of finding the signed chromatic number of a signed interval graph is NP-complete. We also prove that the problem of finding a maximum S-clique in signed interval graphs is in P. We also consider some other related problems.

2 Preliminaries

A graph G of order n is said to have a *perfect elimination ordering* (abbreviated P.E.O) if there is an ordering v_1, v_2, \dots, v_n of vertices of G , such that the sub graph of G induced on the vertex set $\{v_1, \dots, v_i\} \cap N_G[v]$ is a complete graph. It is well known that any interval graph I admits a P.E.O, see for instance [7]. Using this ordering, the vertices of an interval graph I can be colored with w (clique number of I) colors by a greedy algorithm efficiently. Hence the coloring problem of interval graphs is in P, see [4].

A perfect elimination ordering of vertices of an interval graph can be obtained by the following property of its maximal cliques, which is established in [1].

Lemma 2 The graph I is an interval graph if and only if the maximal cliques of it can be ordered M_1, M_2, \dots, M_k such that for any i, j, k , where $i \leq j \leq k$, it has the property $M_i \cap M_k \subseteq M_j$.

In [3] an equivalent condition for a signed graph to be an S-clique is established.

Lemma 3 A signed graph is an S-clique if and only if for each pair u and v of vertices either uv is an edge or u and v are vertices of an unbalanced cycle of length 4.

We define a neighborhood vector of a vertex in a signed graph as follows. Let Σ be a signed graph on the ordered vertex set $V = \{v_1, v_2, \dots, v_n\}$ and $1 \leq i_1 < i_2 < \dots < i_s \leq n$ be an increasing sequence of integers. For $S = \{v_{i_1}, v_{i_2}, \dots, v_{i_s}\}$ and a vertex v of Σ we

define the S -neighborhood vector of v , denoted by $\vec{N}_S(v)$, as bellow.

$$\vec{N}_S(v)_j = \begin{cases} 1 & \text{if there is a positive edge between } v \text{ and } v_{i_j}, \\ -1 & \text{if there is a negative edge between } v \text{ and } v_{i_j}, \\ 0 & \text{otherwise.} \end{cases}$$

The V -neighborhood vector of a vertex v is simply denoted by $\vec{N}(v)$. By the above definition, Lemma 3 can be restated.

Lemma 4 A signed graph on an ordered set V is an S-clique if and only if for each pair u and v of vertices either uv is an edge or for $S = N(u) \cap N(v)$, $\vec{N}_S(u) \neq \pm \vec{N}_S(v)$.

There is a re-formulation for the notion of signed chromatic number in [3] by the means of signed graph homomorphism which follows. A signed graph $[G, \sigma]$ is said to be homomorphic to $[H, \sigma_1]$ if there is an equivalent signified graph (G, σ') of (G, σ) and a mapping $\phi : V(G) \rightarrow V(H)$ such that (i) if $xy \in E(G)$ then $\phi(x)\phi(y) \in E(H)$ and (ii) $xy \in \sigma'$ if and only if $\phi(x)\phi(y) \in \sigma_1$. The chromatic number of a signed graph $[G, \sigma]$ can then be defined as the smallest order of a homomorphic image of $[G, \sigma]$.

Lemma 5 A signified graph $\Sigma = (G, \sigma)$ is balanced if and only if σ is an edge cut of G .

Proof. Since Σ is balanced then there is a subset S of vertices of Σ such that resigning at S leads to (G, \emptyset) . Note that resigning at S does not change the sign of edges with end points in S or \bar{S} , but it changes the sign of all the edges in the edge cut $[S, \bar{S}]$. Hence $\sigma = [S, \bar{S}]$ and the assertion follows. \square

In the following theorem we preview the complexity of some known problems from literature which will be useful later.

Theorem 1

- The maximum independence number of a bipartite graph can be found in a polynomial time, see [6].

- The maximum clique problem in graphs is NP-complete, see [6].
- The biclique problem in bipartite graphs is NP-complete, see [5].
- The problem of finding a maximum edge cut in a graph is NP-complete, see [6].

3 Main results

In this section we present our main results on the complexity of coloring, S-clique problem of signed interval graphs and also some other related problems.

Lemma 6 The maximum S-clique number of a signed interval graph with two maximal cliques can be found in a polynomial time.

Proof. Let I be an interval graph with two maximal cliques. Suppose the the vertices of I have the perfect elimination ordering $\{v_1, v_2, \dots, v_n\}$. Let $\mathcal{I} = (I, \sigma)$ be a signified graph. We may divide the vertices of I to the two subsets $V_1 = \{v_1, \dots, v_s\}$ and $V_2 = \{v_r, \dots, v_n\}$ so that the sub graphs of I induced on any of the subsets V_1 and V_2 are maximal cliques. It is possible by considering s to be the maximum between indices of neighbors of v_1 and r to be the minimum between indices of neighbors of v_n . Indeed $r \leq s$ and any vertex in the set $A = \{v_r, \dots, v_s\}$ is adjacent to all the other vertices in $V(\mathcal{I})$. By Lemma 4, any two vertices in an S-clique of \mathcal{I} should be either adjacent or $\vec{N}_A(u) \neq \pm \vec{N}_A(v)$. We correspond a graph $G = G(\mathcal{I})$ with vertex set $V_1 \Delta V_2$ (the symmetric difference of V_1, V_2) where two vertices are adjacent if and only if they are adjacent in \mathcal{I} or $\vec{N}_A(u) \neq \pm \vec{N}_A(v)$. Suppose that U is the vertex set of a maximum S-clique of \mathcal{I} . Indeed $A \subseteq U$. The vertex set $U \setminus A$ induces a maximum clique in G and any maximum clique in G together with A induces an S-clique in \mathcal{I} . Hence it is enough to find a maximum clique in G . But the graph G has a bipartite complement, and by Theorem 1, the maximum independence number of the complement of G can be determined in a polynomial time this means that a maximum clique in G can be found in polynomial time. \square

Theorem 2 The problem of finding the maximum S-clique number of a signed interval graph \mathcal{I} is in P.

Proof. Let I be an interval graph and \mathcal{I} be a signed graph on it. We prove the assertion by induction on the number of maximal cliques of I . The assertion already follows for the interval graphs with two maximal cliques. Now assume that the interval graph has $k \geq 2$ maximal cliques. Let M_1, M_2, \dots, M_k be the ordering of maximal cliques of I as mentioned in Lemma 2. We correspond a graph G to the signed interval graph \mathcal{I} as follows. The vertex set of G is same as the vertex set of \mathcal{I} where two vertices are adjacent in G if they are adjacent in \mathcal{I} or they are the nodes of an unbalanced 4-cycle in \mathcal{I} . By Lemma 3 signed cliques of \mathcal{I} are in correspondence with cliques of G . Hence it is enough to consider the clique problem for G , or equivalently the maximum independent set problem of G^c . Note that two vertices u, v are adjacent in G^c only if they are non-adjacent in \mathcal{I} and if $u \in M_i$ and $v \in M_j$, where i, j are minimum possible such indices, then $\vec{N}_{M_i \cap M_j}(u) = \pm \vec{N}_{M_i \cap M_j}(v)$. We claim that for any component of G^c say H the induced sub-graph of H on the subset $M_1 \cup M_k$ is a complete bipartite graph. Suppose that the vertices $u \in M_1$ and $v \in M_k$ are connected by a minimum path $u = u_1, u_2, \dots, u_s = v$ in G^c . Note that $u_i \in M_i$, for $i = 1, 2, \dots, s$. By definition of G^c we conclude that $\vec{N}_{M_i \cap M_{i+1}}(u_l) = \pm \vec{N}_{M_i \cap M_{i+1}}(u_{l+1})$ for $l = 0, 1, \dots, s-1$. By Lemma 2 we have $M_1 \cap M_k \subseteq M_i \cap M_j$, for any $1 \leq i \leq j \leq k$ hence $\vec{N}_{M_1 \cap M_k}(u) = \pm \vec{N}_{M_1 \cap M_k}(v)$ so u is adjacent to v and the claim follows. This implies that a maximum independent set in any component of G^c either have empty intersection with M_1 or M_k . Hence a maximum independent set of any component of G^c belongs to $M_2 \cup M_3 \cup \dots \cup M_k$ or $M_1 \cup M_2 \cup \dots \cup M_{k-1}$. Therefore the maximum S-clique of \mathcal{I} can be found by considering a signed interval graph with less than k cliques. Hence by induction assumption the assertion follows. \square

The proofs of the above lemmas can be used to construct a maximum S-clique for signed interval graphs.

Theorem 3 The coloring problem of signed interval graphs is NP-complete.

Proof. Let $n > 1$ be an integer and I be the interval graph on the following list of intervals,

$$A_i = [\frac{i}{2n}, 3 - \frac{i}{2n}], i = 1, \dots, n.$$

$$B_i = [\frac{1}{2} + \frac{i}{2n}, \frac{3}{2} - \frac{i}{2n}], i = 1, \dots, n.$$

$$C_i = [\frac{3}{2} + \frac{i}{2n}, \frac{5}{2} - \frac{i}{2n}], i = 1, \dots, n.$$

We consider the following labeling of vertices.

$$l(B_i) = i, i = 1, 2, \dots, n.$$

$$l(A_i) = n + i, i = 1, 2, \dots, n.$$

$$l(C_i) = 2n + i, i = 1, 2, \dots, n.$$

Let H be the sub set of vertices labeled by $1, 2, \dots, 2n$ and K be the sub set of vertices labeled by $2n + 1, 2n + 2, \dots, 3n$. Note that the sub graphs $\langle H \rangle$ and $\langle K \rangle$ are complete graphs. Let σ be a subset of edges of I such that both ends belong to K . Consider the signed graph $\mathcal{I} = [I, \sigma]$. Suppose that the vertices of \mathcal{I} are properly colored with $\chi_s(\mathcal{I})$ colors. We may suppose that vertices of H are colored with $\{1, 2, \dots, 2n\}$, as $\langle H \rangle$ is a complete graph. Now suppose that L is the set of those vertices in K which are colored by any of colors in $\{1, 2, \dots, 2n\}$. It is clear that the set L has maximum possible size considering different proper colorings of \mathcal{I} . Let C be the set of vertices of \mathcal{I} indexed by $n + 1, n + 2, \dots, 2n$. Any proper coloring of \mathcal{I} should have the following properties.

- The colors of vertices in C can not be used to color any vertex in K .
- If $v \in L$, then $\vec{N}_C(v) = \pm j_n$, where j_n is the vector all of it's entries equal 1.
- If $u, v \in L$, then one of the followings hold:
 - $\vec{N}_C(u) = \vec{N}_C(v)$ and the edge between u, v in \mathcal{I} is positive.
 - $\vec{N}_C(u) = -\vec{N}_C(v)$ and the edge between u, v in \mathcal{I} is negative.

Let X be the set of vertices in K , say v , such that $\vec{N}_C(v) = j_n$ and Y be the set of vertices in K such that $\vec{N}_C(v) = -j_n$. Let L_1 be $L \cap X$ and L_2 be $L \cap Y$. Let $\langle X, Y \rangle^-$ be the signed sub graph of \mathcal{I} with vertex set $X \cup Y$ where the edges are the set of negative edges of \mathcal{I} with one end in X and the other end in Y . As discussed already we know that the sub graphs of \mathcal{I} induced on the sets L_1 and L_2 are positive and all the edges between L_1 and L_2 are negative. Any of the following cases may occur.

- The set L_1 is empty. In this case the set L_2 induces a maximum clique in the graph $\langle Y \rangle^+$.

- The set L_2 is empty. In this case the set L_1 induces a maximum clique in the graph $\langle X \rangle^+$.
- Neither L_1 nor L_2 are empty. In this case the bipartite signed sub graph of \mathcal{I} induced on the partitions L_1, L_2 is the maximum biclique of the graph $\langle X, Y \rangle^-$.

Hence the coloring problem in the graph \mathcal{I} is equivalently one of the problems of finding clique number of one of the graphs $\langle Y \rangle^+, \langle X \rangle^+$ or finding a maximum biclique number in the graph $\langle X, Y \rangle^-$. Since the stated graphs are arbitrary thus by Theorem 1, the coloring problem of the signed interval graphs is NP-complete. \square

4 Miscellaneous

In this section we present some minor results.

Proposition 1 Let (G, σ_0) be a signified graph. Then the problem of finding a minimum set σ such that (G, σ) and (G, σ_0) are switching equivalent, is an NP-complete problem.

Proof. Let $(G, \sigma_0), (G, \sigma)$ be as desired. Suppose that the signified graph (G, σ) is obtained from (G, σ_0) by resigning at the vertices in $S \subseteq V(G)$. Resigning of the vertices in S does not change the sign of the edges in the sub graphs $\langle S \rangle$ and $\langle \overline{S} \rangle$ of G . But it changes the sign of all the edges in the edge cut $[S, \overline{S}]$. Hence we have to choose the subset S in such a way that there exist maximum possible negative edges in the edge cut $[S, \overline{S}]$. Hence $[S, \overline{S}]$ should be a maximum edge cut in the graph Σ^- . Since the signature and hence the graph Σ^- are arbitrary, the assertion follows by Theorem 1. \square

Proposition 2 Let $\mathcal{I} = [I, \sigma]$ be a signed interval graph. Then the followings hold,

- $\chi_s(\mathcal{I}) = 2$, if and only if I is a tree.
- $\chi_s(\mathcal{I}) = 3$, then I is an interval graph with clique number 3 and all the triangles have the same signature.

Proof. The proof is straightforward. \square

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